First-Order Differential Equations Review

We consider first-order differential equations of the form:

$$\frac{dx(t)}{dt} + \frac{1}{\tau} x(t) = f(t) \qquad (1)$$

where $f(t)$ is the forcing function. In general, the differential equation has two solutions:

1. *complementary* (or natural or homogeneous) solution, $x_C(t)$ (when $f(t) = 0$), and
2. *particular* (or forced or non-homogeneous) solution, $x_P(t)$ (when $f(t) \neq 0$).

In our problems, $f(t)$ is often a constant, and therefore, the overall solution to the differential equation is

$$x(t) = x_C(t) + x_P(t) = K_1 \, e^{-t/\tau} + K_2 \qquad (2)$$

The *time constant*, τ, for first-order electrical circuits is either $\tau = RC$ or $\tau = L/R$.

Complementary Solution

The complementary solution is found by considering the *homogeneous equation*:

$$\frac{dx(t)}{dt} + \frac{1}{\tau} x(t) = 0 \qquad (3)$$

The complementary solution is the system's *natural response*, which is

$$x_C(t) = K_1 \, e^{-t/\tau}$$

This result can be verified by substituting this answer into the differential equation, Eq. (3).

Particular Solution

The particular solution is found by considering the full (non-homogeneous) differential equation, that is, Eq. (1). If the forcing function is a constant, then $x_P(t)$ is a constant (K_2) also, and hence $dx_P/dt = 0$. Substituting x_P into the original differential equation, Eq. (1), yields:

$$\frac{dx_P}{dt} + \frac{1}{\tau} x_P = 0 + \frac{1}{\tau} x_P = f$$

Reducing the above expression provides the particular solution, which is the *forced response*:

$$x_P = K_2 = \tau \, f$$

Note that, alternatively, K_2 could have been found using Eq. (2) while considering the final value reached by the variable of interest, and hence, in this case is also the *steady-state response*:

$$x(\infty) = K_1 \, e^{-\infty} + K_2 = K_2$$

Total Solution

The total solution is the sum of the complementary and particular solutions:

$$x_T(t) = x_C(t) + x_P = K_1 \, e^{-t/\tau} + \tau \, f \qquad (4)$$

The coefficient term, K_1, is found from the initial conditions (at $t = 0$).

Initial Conditions

Typically, the initial conditions are determined by considering that for dc steady-state conditions: (*a*) a capacitor is like an *open circuit* ($i_C = C \, dv_C/dt = 0$), and (*b*) an inductor appears as a *short circuit* ($v_L = L \, di_L/dt = 0$). Further, the current through an inductor and the voltage across a capacitor cannot change instantaneously, for example, $i_L(0-) = i_L(0+)$ and $v_C(0-) = v_C(0+)$.

Second-Order Differential Equations Review

The second-order differential equations of interest are of the form:

$$\frac{d^2 y}{dt^2} + 2\varsigma\,\omega_0 \frac{d\,y}{dt} + \omega_0^2\, y(t) = f(t) \qquad (5)$$

where $f(t)$ is the forcing function. Like the first-order case, this differential equation has two solutions: (1) *complementary* (or natural) solution, $y_C(t)$ when $f(t) = 0$, and (2) *particular* (or forced) solution, $y_P(t)$ when $f(t) \neq 0$. In our problems, $f(t)$ is often a constant, and therefore, the overall solution to the differential equation is typically

$$y(t) = y_C(t) + y_P(t) = K_1\, e^{s_1 t} + K_2\, e^{s_2 t} + K_3 \qquad (6)$$

<u>Complementary Solution</u>

The complementary solution is found by considering the homogeneous equation:

$$\frac{d^2 y}{dt^2} + a\frac{d\,y}{dt} + b\,y(t) = 0 \qquad (7)$$

where $a = 2\varsigma\,\omega_0$ and $b = \omega_0^2$. The complementary solution is the system's *natural response*:

$$y_C(t) = K_1\, e^{s_1 t} + K_2\, e^{s_2 t} \qquad (8)$$

The *natural frequencies*, s_1 and s_2, are the roots of the *characteristic equation*:

$$s^2 + a\,s + b = 0 \qquad (9)$$

These roots lead to three possible cases:

ζ	Roots (s_1, s_2)	Damping	Natural Response
$\zeta > 1$	Real and unequal	*Overdamped*	Sum of two decaying exponentials
$\zeta = 1$	Real and equal	*Critically damped*	Two faster decaying exponentials
$\zeta < 1$	Complex conjugates	*Underdamped*	Exponentially damped sinusoid

<u>Particular Solution</u>

The particular solution is found by considering the full (non-homogeneous) differential equation, Eq. (5). If the forcing function is a constant, then $y_P(t)$ is also a constant (K_3), and hence $dy_P/dt = 0 = d^2 y_P/dt^2$. Substituting y_P into the differential equation yields:

$$\frac{d^2 y}{dt^2} + a\frac{d\,y}{dt} + b\,y(t) = 0 + a\;0 + b\;y_P = f$$

Reducing the above expression provides the particular solution, which is the *forced response*

$$y_P = K_3 = \frac{f}{b}$$

Alternatively, K_3 could have been determined from Eq. (6) and the final value $y(\infty)$.

<u>Initial Conditions</u>

Initial conditions for y and its derivative are necessary to find K_1 and K_2; specifically needed are $y(0)$ and $\left.\frac{dy}{dt}\right|_{t=0}$. A numerical value for $\left.\frac{dy}{dt}\right|_{t=0}$ is generally found by re-applying KVL or KCL at t=0. Practically speaking, as the KVL or KCL is performed, the inductor voltage and capacitor current, respectively, are expressed as $\left.L\,di_L/dt\right|_{t=0}$ and $\left.C\,dv_C/dt\right|_{t=0}$. The numerical value of $\left.\frac{dy}{dt}\right|_{t=0}$ is then equated to an analytical derivative of $y(t)$ determined from Eq. (6). The initial conditions for $y(0)$ are found in the same manner as described for first-order differential equations.