RELATIVITY

Relativistic Mass

The relativistic mass (m) is found from Einstein's *special theory of relativity*

$$m = \frac{m_0}{\sqrt{1-(v^2/c^2)}} \tag{1}$$

where the subscript "0" denotes zero velocity, v (*i.e.*, at rest) and c is the speed of light.

Rest Mass Energy

For particles with mass (such as electrons, protons, neutrons, alpha particles, etc.), the rest mass energy is

$$E_{rest} = m_0\, c^2 \tag{2}$$

where m_0 is the mass that is given in reference tables. Note that photons, such as gamma and X rays, have no rest mass energy since they are pure electromagnetic energy without mass.

Total Energy

The total energy (E_{total}) is the sum of the rest mass and kinetic energies (but does not generally include any potential energy for the purposes here). For relativistic particles (*e.g.*, fast electrons), their total energy is:

$$E_{total} = E_{rest} + E_{kinetic} = m\,c^2 = \frac{m_0\, c^2}{\sqrt{1-(v^2/c^2)}} \tag{3}$$

The above formulation can be used for any particle; however, for particle velocities that are not near the speed of light ($<0.1c$), the simpler classical mechanics expression is applicable. That is, for non-relativistic particles (*e.g.*, most neutrons), Equation (3) simplifies to

$$E_{total} = E_{rest} + E_{kinetic} = m_0\, c^2 + \tfrac{1}{2} m_0\, v^2 \tag{4}$$

Generally, the classical formula can be used for heavy particles (*i.e.*, of proton mass or larger).

The total energy of a photon, which moves at the speed of light ($c = \lambda\, f$), is

$$E_{total} = h\, f = \frac{h\,c}{\lambda} \tag{5}$$

where h is Planck's constant, f is the frequency, and λ is the *wavelength*.

Kinetic Energy

The kinetic energy ($E_{kinetic}$) is the energy associated with the fact that the particle is moving. When a particle is described as being of a certain energy, it is the kinetic energy to which is being referred; for example, a 2 MeV neutron has a kinetic energy of 2 MeV. For relativistic particles (*e.g.*, fast electrons), we use

$$E_{kinetic} = E_{total} - E_{rest} = m\,c^2 - m_0\, c^2 = m_0\, c^2 \left[\frac{1}{\sqrt{1-(v^2/c^2)}} - 1 \right] \tag{6}$$

For non-relativistic particles, *i.e.*, for $v << c$, (*e.g.*, most heavy particles) the above expression reduces to the classic formula:

$$E_{kinetic} = \tfrac{1}{2} m_0\, v^2 \tag{7}$$

Since photons are of zero rest mass and kinetic energy is a consequence of particle motion, Eq. (6) implies that the kinetic and total energies of a photon are identical.

Momentum and Wavelength

For cases of both relativistic and classical mechanics, the momentum is

$$p \equiv m\,v \tag{8}$$

The momentum and wavelength are interrelated quantities via

$$\lambda = h/p \tag{9}$$

For relativistic particles (*e.g.*, fast electrons), various formula can be written for these quantities

$$\begin{aligned} p = m\,v = \frac{E_{total}\,v}{c^2} &= \frac{m_0\,v}{\sqrt{1-(v^2/c^2)}} \\ = \frac{1}{c}\sqrt{E_{total}^2 - E_{rest}^2} &= \frac{1}{c}\sqrt{E_{kinetic}^2 + 2\,E_{kinetic}\,E_{rest}} \end{aligned} \tag{10}$$

where the latter expressions are obtained by beginning with $p^2 = (m\,v)^2$

$$\lambda = \frac{h}{p} = \frac{h\,c}{\sqrt{E_{total}^2 - E_{rest}^2}} \tag{11}$$

For non-relativistic particles (*e.g.*, heavy particles), the momentum and wavelength are simply

$$p = m\,v = \sqrt{2\,m_0 E_{kinetic}} \tag{12}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2\,m_0\,E_{kinetic}}} \tag{13}$$

For particles of zero rest mass (*e.g.*, photons), we assume a relativistic mass

$$p = m\,v = \frac{E\,v}{c^2} = \frac{E}{c} = \frac{h\,f}{c} = \frac{h}{\lambda} \tag{14}$$

$$\lambda = \frac{h}{p} = \frac{h\,c}{E} = \frac{c}{f} \tag{15}$$

Example: Compute the frequency, wavelength and momentum of a 1 MeV X-ray.

Solution: Using Eq. (5), the photon frequency and wavelength may be calculated:

$$f = \frac{E}{h} = \frac{1\text{ MeV}}{4.1356673\times10^{-21}\text{ MeV}\cdot\text{sec}} = 2.42\times10^{20}\text{ Hz}$$

$$\lambda = \frac{c}{f} = \frac{2.998\times10^{8}\text{ m/sec}}{2.42\times10^{20}\text{ Hz}} = 1.24\times10^{-12}\text{ m} = 1.24\text{ pm} = 0.0124\,\text{Å}$$

The photon momentum is found from Eq. (14):

$$p = \frac{E}{c} = \frac{(1\text{ MeV})(1.6022\times10^{-13}\text{ J/MeV})}{(2.998\times10^{8}\text{ m/sec})(\text{J}/\frac{\text{kg}\cdot\text{m}^2}{\text{sec}^2})} = 5.344\times10^{-22}\ \frac{\text{kg}\cdot\text{m}}{\text{sec}}$$