SOLAR CALCULATIONS

The orbit of the Earth is an ellipse not a circle, hence the distance between the Earth and Sun varies over the year, leading to apparent solar irradiation values throughout the year approximated by [1]:

$$I_0 = I_{SC}\left[1 + 0.033\cos\left(\frac{N}{365}\times 360°\right)\right] \qquad (1)$$

where the *solar constant*, I_{SC} = 429.5 Btu/hr·ft² (1353 W/m²). The Earth's closest point (about 146 million km) to the sun is called the *perihelion* and occurs around January 3; the Earth's farthest point (about 156 million km) to the sun is called the *aphelion* and occurs around July 4.

The Earth is tilted on its axis at an angle of 23.45°. As the Earth annually travels around the sun, the tilting manifests itself as our seasons of the year. The sun crosses the equator around March 21 (vernal equinox) and September 21 (autumnal equinox). The sun reaches its northernmost latitude about June 21 (summer solstice) and its southernmost latitude near December 21 (winter solstice).

The declination is the angular distance of the sun north or south of the earth's equator. The d*eclination angle*, δ, for the Northern Hemisphere (reverse the declination angle's sign for the Southern Hemisphere) is [2]

$$\delta = 23.45°\sin\left[\frac{N+284}{365}\times 360°\right] \qquad (2)$$

where *N* is the day number of the year, with January 1 equal to 1.

The Earth is divided into latitudes (horizontal divisions) and longitudes (N-S divisions). The equator is at a latitude of 0°; the north and south poles are at +90° and –90°, respectively; the Tropic of Cancer and Tropic of Capricorn are located at +23.45° and –23.45°, respectively. For longitudes, the global community has defined 0° as the prime meridian which is located at Greenwich, England. The longitudes are described in terms of how many degrees they lie to the east or west of the prime meridian. A 24-hr day has 1440 mins, which when divided by 360°, means that it takes 4 mins to move each degree of longitude.

The apparent solar time, *AST* (or local solar time) in the western longitudes is calculated from

$$AST = LST + (4\ \text{min/deg})(LSTM - Long) + ET \qquad (3)$$

where

LST = Local standard time or clock time for that time zone (may need to adjust for daylight savings time, *DST*, that is *LST* = *DST* – 1 hr),
Long = local longitude at the position of interest, and
LSTM = local longitude of standard time meridian

$$LSTM = 15° \times \left(\frac{Long}{15°}\right)_{\text{round to integer}} \qquad (4)$$

The difference between the true solar time and the mean solar time changes continuously day-to-day with an annual cycle. This quantity is known as the *equation of time*. The equation of time, *ET* in minutes, is approximated by [3]

$$ET = 9.87\sin(2D) - 7.53\cos(D) - 1.5\sin(D)$$
$$\text{where} \quad D = 360°\,\frac{(N-81)}{365} \tag{5}$$

Example 1: Find the *AST* for 8:00 a.m. MST on July 21 in Phoenix, AZ, which is located at 112° W longitude and a northern latitude of 33.43°.
Solution: Since Phoenix does not observe daylight savings time, it is unnecessary to make any change to the local clock time. Using Table I, July 21 is the 202nd day of the year. From Equation (5), the equation of time is $D = 360°\,\frac{(N-81)}{365} = 360°\,\frac{(202-81)}{365} = 119.3°$; $ET = 9.87\sin(2D) - 7.53\cos(D) - 1.5\sin(D) = 9.87\sin(2\times 119.3°) - 7.53\cos(119.3°) - 1.5\sin(119.3°) = -6.05\text{ min}$. From Equation (4), the local standard meridian for Phoenix is $LSTM = 15° \times \left(\frac{Long}{15°}\right)_{\text{round to integer}} = 15° \times \left(\frac{112°}{15°}\right)_{\text{round to integer}} = 15° \times 7 = 105°$. Using Equation (3), the apparent solar time (*AST*) is $AST = LST + (4\text{ mins})(LSTM - Long) + ET = 8:00 + (4\text{ mins})(105° - 112°) + (-6.05\text{ min}) = 7:26\text{ a.m.}$

The *hour angle*, *H*, is the azimuth angle of the sun's rays caused by the earth's rotation, and *H* can be computed from [4]

$$H = \frac{(No.\ of\ minutes\ past\ midnight, AST) - 720\text{ mins}}{4\text{ min/deg}} \tag{6}$$

The hour angle as defined here is negative in the morning and positive in the afternoon ($H = 0°$ at noon).

The solar *altitude angle* (β_1) is the apparent angular height of the sun in the sky if you are facing it. The *zenith angle* (θ_z) and its complement the altitude angle (β_1) are given by

$$\cos(\theta_z) = \sin(\beta_1) = \cos(L)\cos(\delta)\cos(H) + \sin(L)\sin(\delta) \tag{7}$$

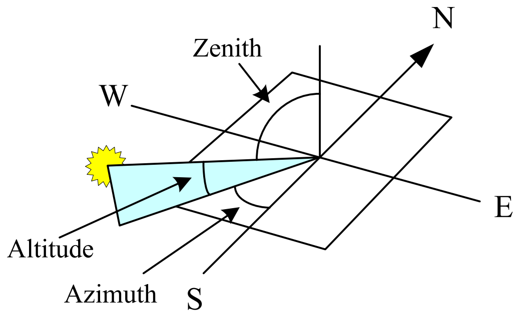
where

L = latitude (positive in either hemisphere) [0° to +90°],
δ = declination angle (negative for Southern Hemisphere) [–23.45° to +23.45°], and
H = hour angle.

The noon altitude is $\beta_N = 90° - L + \delta$. The sun rises and sets when its altitude is 0°, not necessarily when its hour angle is ±90°. The hour angle at sunset or sunrise, H_S, can be found from using Eq. (7) when $\beta_1 = 0$

$$\cos(H_S) = -\tan(L)\tan(\delta) \tag{8}$$

where H_S is negative for sunrise and positive for sunset. Absolute values of $\cos(H_S)$ greater than unity occur in the arctic zones when the sun neither rises nor sets.



The *solar azimuth*, α_1, is the angle away from south (north in the Southern Hemisphere).

$$\cos(\alpha_1) = \frac{\sin(\beta_1)\sin(L) - \sin(\delta)}{\cos(\beta_1)\cos(L)} \tag{9}$$

where α_1 is positive toward the west (afternoon), and negative toward the east (morning), and therefore, the sign of α_1 should match that of the hour angle.

If $\delta > 0$, the sun can be north of the east-west line. The time at which the sun is due east and west can be determined from

$$t_{E/W} = 4\frac{\text{min}}{\text{deg}}\left\{180° \mp \arccos\left[\frac{\tan(\delta)}{\tan(L)}\right]\right\} \tag{10}$$

where these times are given in minutes from midnight AST.

Example 2: For the conditions of Example 1, determine the solar altitude and azimuth angles.

Solution: The hour angle is first computed using Equation (6)

$$H = \frac{(No.\ of\ \ minutes\ \ past\ midnight, AST) - 720\text{ mins}}{4\text{ min/deg}} = \frac{[60*7+26]-720\text{min}}{4\text{ min/deg}} = -68.5°$$

The declination angle is found from Equation (2)

$$\delta = 23.45°\sin\left[\frac{N+284}{365}\times 360°\right] = 23.45°\sin\left[\frac{202+284}{365}\times 360°\right] = 20.44°$$

The altitude angle (β_1) of the sun is calculated via Equation (7)

$$\sin(\beta_1) = \cos(L)\cos(\delta)\cos(H) + \sin(L)\sin(\delta)$$

$$\beta_1 = \arcsin[\cos(33.43°)\cos(20.44°)\cos(-68.5°) + \sin(33.43°)\sin(20.44°)] = 28.62°$$

The solar azimuth angle (α_1) is found from Equation (9)

$$\alpha_1 = \arccos\left[\frac{\sin(\beta_1)\sin(L) - \sin(\delta)}{\cos(\beta_1)\cos(L)}\right][\mathrm{sgn}(H)]$$

$$= \arccos\left[\frac{\sin(28.62°)\sin(33.43°) - \sin(20.44°)}{\cos(28.62°)\cos(33.43°)}\right]\mathrm{sgn}(-68.5°) = -96.69°$$

This value indicates that the sun is north of the east-west line. The time at which the sun is directly east can be computed using Equation (10)

$$t_E = 4\,\mathrm{min}\left\{180° - \arccos\left[\frac{\tan(\delta)}{\tan(L)}\right]\right\} = 4\,\mathrm{min}\left\{180° - \arccos\left[\frac{\tan(20.44°)}{\tan(33.43°)}\right]\right\} = 8:17.5 \text{ a.m.}$$

The AST is earlier than t_E, thus verifying that the sun is still north the east-west line.

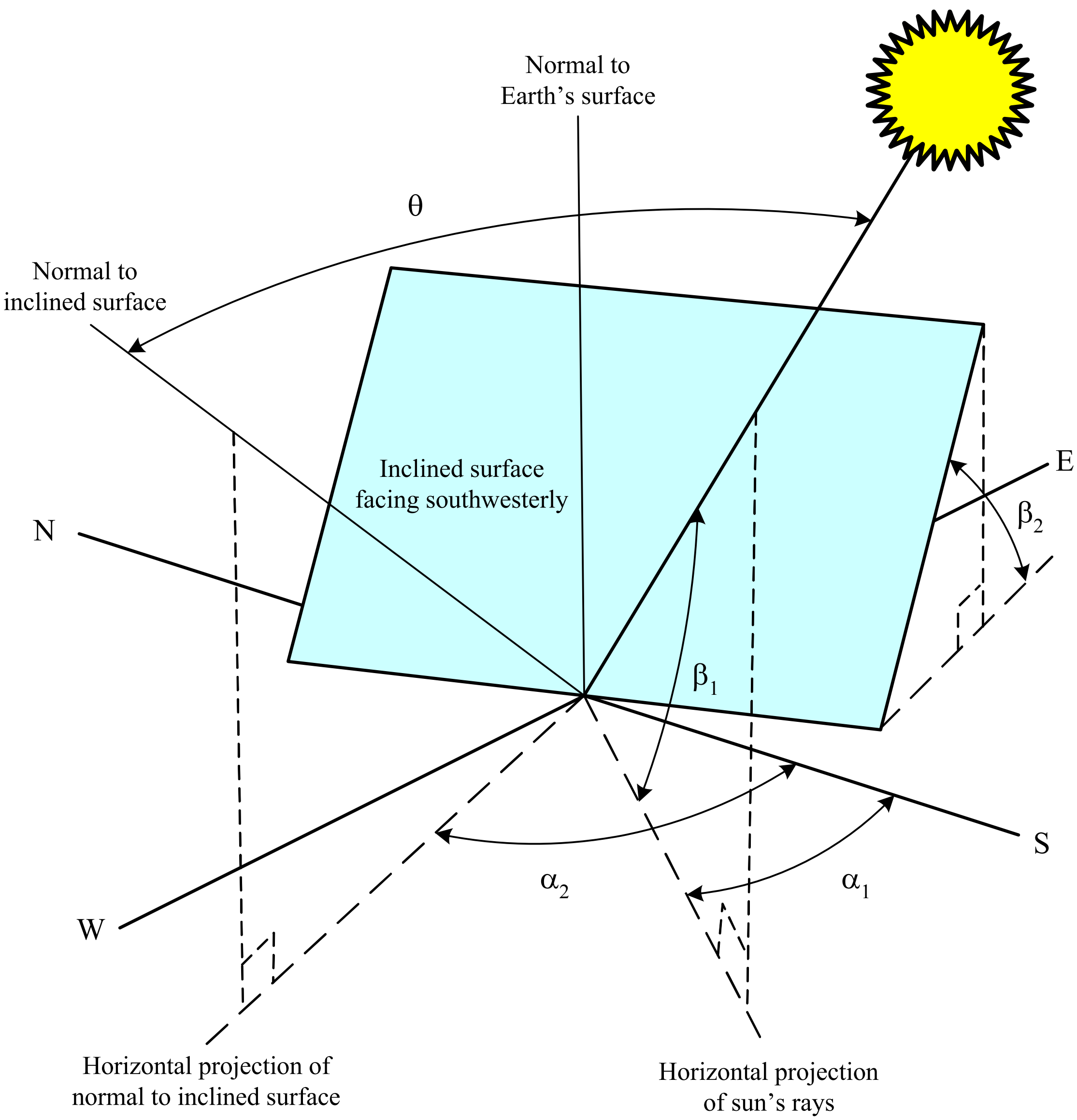


Figure 1. Solar angles [4]

The collector angle (θ) between the sun and normal to the surface is

$$\cos(\theta) = \sin(\beta_1)\cos(\beta_2) + \cos(\beta_1)\sin(\beta_2)\cos(\alpha_1 - \alpha_2) \qquad (11)$$

where α_2 is the azimuth angle normal to the collector surface, and β_2 is the tilt angle from the ground. If θ is greater than 90°, then the sun is behind the collector. Some collector panel angles of interest are given below.

Azimuth, α_2	Tilt, β_2	θ	Orientation
n/a	0°	90°–β_1	Horizontal (flat)
–	90°	varies	Vertical wall
0°	90°	varies	Southfacing Vertical
–90°	90°	varies	East facing wall
+90°	90°	varies	West facing wall
α_1	90°–β_1	0°	Tracking System

The sunrise and sunset hours on the collector are different when the collector is shadowed by itself. The collector panel sunrise/sunset hours may be computed from [1]

$$H_{SRS} = \mp \min\left\{ H_S, \arccos\left[\frac{a\,b \pm \sqrt{a^2 - b^2 + 1}}{a^2 + 1} \right] \right\} \qquad (12)$$

where

$$a = \frac{\cos(L)}{\sin(\alpha_2)\tan(\beta_2)} + \frac{\sin(L)}{\tan(\alpha_2)}$$
$$b = \tan(\delta)\left[\frac{\cos(L)}{\tan(\alpha_2)} - \frac{\sin(L)}{\sin(\alpha_2)\tan(\beta_2)} \right] \qquad (13)$$

Example 3: For the conditions of Example 1, find the collector angle for a wall that faces east-southeast and is tilted at an angle equal to the location latitude.

Solution: The latitude is 33.43° which is also the tilt angle (β_2). Referring to Figure 1, the collector azimuth angle (α_2) for an east-southeast direction is $-90° \times 3/4 = -67.5°$. Finally, the collector angle is computed from Equation (11)

$$\theta = \arccos[\sin(\beta_1)\cos(\beta_2) + \cos(\beta_1)\sin(\beta_2)\cos(\alpha_1 - \alpha_2)]$$
$$= \arccos[\sin(28.62°)\cos(33.43°) + \cos(28.62°)\sin(33.43°)\cos(-96.69° - (-67.5°))] = 34.7°$$

Example 4: Determine the time of sunrise for the conditions of the previous examples.

Solution: Using Equation (8), we find the (negative) hour angle for sunrise is

$$H_S = -\arccos[-\tan(L)\tan(\delta)] = -\arccos[-\tan(33.43°)\tan(20.44°)] = -104.2°$$

To find the corresponding sunrise time in AST, we rearrange Equation (6)

$$(\mathit{Sunrise}, \mathit{AST}) = 720 \text{ mins} + H_S(4 \text{ min/deg}) = 720 + (-104.2°)(4\,\tfrac{\text{min}}{\text{deg}}) = 5:03 \text{ a.m.}$$

The corresponding local time is then found using Equation (3).

$$LST = AST - (4 \text{ mins})(LSTM - Long) - ET$$
$$= 5:03 \text{ a.m.} - (4 \text{ mins})(105° - 112°) - (-6.05 \text{ min}) = 5:37 \text{ a.m.}$$

The direct normal irradiance to the ground is [1]

$$I_{DN} = A \exp\left(-\frac{p}{p_0} \frac{B}{\sin(\beta_1)} \right) \tag{14}$$

where A is the apparent extraterrestrial solar intensity*, B is the atmospheric extinction coefficient (mainly due to changes in atmospheric moisture), and p/p_0 is the pressure at the location of interest relative to a standard atmosphere, given by

$$p/p_0 = \exp(-0.0000361\, z) \tag{15}$$

where z is the altitude in feet above sea level. The direct normal radiation at sea-level then is

$$I_{DN}(0\,\text{ft}) = A \exp\left[\frac{-B}{\sin(\beta_1)} \right] \tag{16}$$

Table I: Apparent solar irradiation (A), Atmospheric extinction coefficient (B), and Ratio of diffuse radiation on a horizontal surface to the direct normal irradiation (C) [2]

Date	Day of Year	A (Btu/hr·ft^2)	B	C
Jan 21	21	390	0.142	0.058
Feb 21	52	385	0.144	0.060
Mar 21	80	376	0.156	0.071
Apr 21	111	360	0.180	0.097
May 21	141	350	0.196	0.121
June 21	172	345	0.205	0.134
July 21	202	344	0.207	0.136
Aug 21	233	351	0.201	0.122
Sept 21	264	365	0.177	0.092
Oct 21	294	378	0.160	0.073
Nov 21	325	387	0.149	0.063
Dec 21	355	391	0.142	0.057

Note: 1 W/m^2 = 0.3173 Btu/hr·ft^2

The direct radiation flux onto the collector is

$$I_D = I_{DN} \cos(\theta) \tag{17}$$

The diffuse-scattered radiation flux onto the collector is

* Page 69 of Ref. 1 describes the procedure for finding A for the Southern Hemisphere.

$$I_{DS} = C\, I_{DN} \left[\frac{1+\cos(\beta_2)}{2}\right] \qquad (18)$$

where *C* is the ratio of diffuse radiation on a horizontal surface to the direct normal irradiation. The reflected radiation flux for a non-horizontal surface may be approximated by

$$I_{DR} = I_{DN}\ \rho\left(C + \sin(\beta_1)\right)\left[\frac{1-\cos(\beta_2)}{2}\right] \qquad (19)$$

where ρ is the foreground reflectivity with some values given below

ρ	Surroundings condition
0.2	Ordinary ground or vegetation
0.8	Snow cover
0.15	Gravel roof

The total radiation flux is then

$$I_{Tot} = I_D + I_{DS} + I_{DR} \qquad (20)$$

Example 5: Determine the direct and diffuse-scattered radiation flux to the collector of Example 3, where Phoenix is at an elevation of 1112 ft.

Solution: The pressure ratio is computed using Equation (15)

$$p/p_0 = \exp(-0.0000361\, z) = \exp(-0.0000361 \times 1112) = 0.9607$$

Extracting the July 21 values of A and B from Table I, and using Equation (14) yields a direct normal radiation of

$$I_{DN} = A \exp\left(-\frac{p}{p_0}\frac{B}{\sin(\beta_1)}\right) = 344\,\tfrac{\text{Btu}}{\text{hr}\cdot\text{ft}^2} \exp\left(-(0.9607)\frac{0.207}{\sin(28.62°)}\right) = 227\,\frac{\text{Btu}}{\text{hr}\cdot\text{ft}^2}$$

From Equation (17), the direct radiation onto the collector is

$$I_D = I_{DN}\cos(\theta) = \left(227\,\tfrac{\text{Btu}}{\text{hr}\cdot\text{ft}^2}\right)\cos(34.7°) = 186.6\,\tfrac{\text{Btu}}{\text{hr}\cdot\text{ft}^2}$$

Using Equation (18), the diffuse scattered radiation flux to the collector is

$$I_{DS} = C\, I_{DN}\left[\frac{1+\cos(\beta_2)}{2}\right] = (0.136)\left(227\,\tfrac{\text{Btu}}{\text{hr}\cdot\text{ft}^2}\right)\left[\frac{1+\cos(33.43°)}{2}\right] = 28.3\,\tfrac{\text{Btu}}{\text{hr}\cdot\text{ft}^2}$$

References

1. P. J. Lunde, *Solar Thermal Engineering: Space Heating and Hot Water Systems*, John Wiley & Sons, 1980, pp. 62-100. [TH7413.L85]
2. Chapter 30, "Solar Energy Utilization," *ASHRAE Handbook, Applications*, 1995. [TH7011.A12]
3. SunAngle, http://www.susdesign.com/sunangle/.
4. A.W. Culp, *Principles of Energy Conversion*, 2nd ed., McGraw-Hill, 1991, pp. 98-107.