THERMAL NEUTRONS

Maxwellian Distribution

The thermal neutron velocity/energy ($E_K = \frac{1}{2} m_n v^2$) distribution is Maxwellian. The number of neutrons of energy E per unit energy interval, $N(E)$, and the number of neutrons of velocity v per unit velocity interval, $N(v)$, can be expressed in terms of the neutron energy:

$$\frac{dN_0}{dE} = N(E) = \frac{2\pi N_0}{(\pi k T)^{3/2}} \sqrt{E}\, e^{-E/kT} \tag{1}$$

or the neutron velocity:

$$\frac{dN_0}{dv} = N(v) = \frac{4\pi v^2 N_0}{(2\pi k T/m)^{3/2}} e^{-mv^2/2kT} \tag{2}$$

where k is Boltzmann's constant; T is the absolute temperature of the medium; and N_0 is the total number of neutrons per unit volume, that is,

$$N_0 = \int_0^\infty N(v)\, dv = \int_0^\infty N(E)\, dE \tag{3}$$

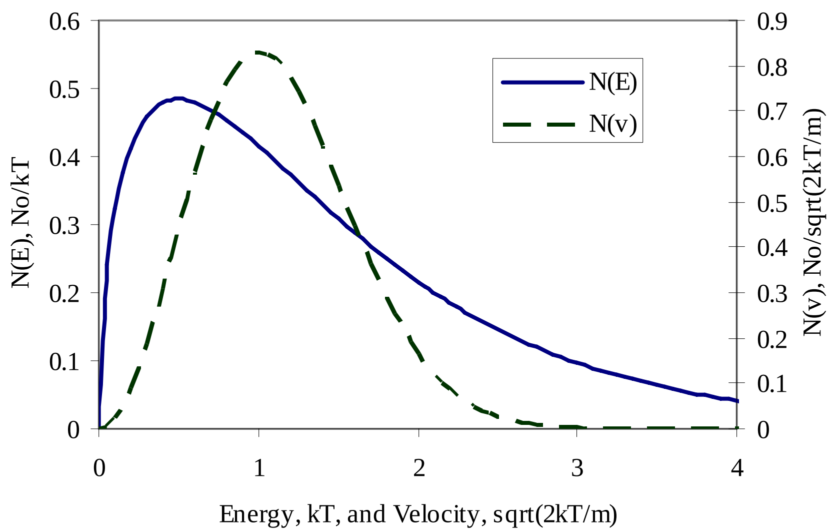

Figure: Maxwellian neutron energy and velocity distributions with normalized units.

By setting the derivative of the $N(E)$ and $N(v)$ expressions equal to zero, the most probable energy and velocity, respectively, can be solved for. We use the product rule to find the derivative of $dN(v)/dv$:

$$\begin{aligned}\frac{dN}{dv} &= \frac{d}{dv}\left[\frac{4\pi v^2 N_0}{(2\pi k T/m)^{3/2}} e^{-mv^2/2kT}\right] \\ &= \frac{4\pi N_0}{(2\pi k T/m)^{3/2}}\left[2v\, e^{-mv^2/2kT} + v^2\, e^{-mv^2/2kT}\left(\frac{-m\,2v}{2kT}\right)\right]\end{aligned} \tag{4}$$

By setting the above expression equal to zero, we find the most probable velocity

$$\frac{dN}{dv} = 0 = 2v + v^2 \left(\frac{-mv}{kT} \right) \qquad (5)$$
$$v_p = \sqrt{\frac{2kT}{m}}$$

This result agrees with the graphical plot of $N(v)$. For a neutron at 20°C, the most probable velocity is then

$$v_0 = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{(2)(1.38\times10^{-23}\ \text{J/°K})(273+20\ \text{°K})}{(1.675\times10^{-27}\ \text{kg})}\left(\frac{\text{kg}\cdot\text{m}^2/\text{s}^2}{\text{J}}\right)} = 2197\ \text{m/sec} \qquad (6)$$

where the zero subscript implies thermal equilibrium at this reference (room) temperature. Because these are low velocity neutrons, the energy at the most probable velocity may be found from the classical expression for kinetic energy

$$E_T = \frac{1}{2} m v_p^2 = \frac{1}{2} m \left(\frac{2kT}{m} \right) = kT \qquad (7)$$

which is the *thermal neutron energy*. At 20° the thermal neutron energy is

$$E_0 = kT = (8.617\times10^{-5}\ \text{eV/°K})(273+20\ \text{°K}) = 0.0253\ \text{eV} \qquad (8)$$

The formulae for the most probable neutron energy, and its corresponding velocity, can be obtained in similar fashion. Left as an exercise, the most probable neutron energy is

$$E_p = \tfrac{1}{2} kT \qquad (9)$$

which agrees with the earlier graph. Note the difference between the most probable energy of Eq. (7), and the energy at the most probable velocity from Eq. (9).

The average energy and velocity can be found from

$$E_{avg} = \frac{\int_0^\infty N(E)\, E\, dE}{\int_0^\infty N(E)\, dE} \qquad v_{avg} = \frac{\int_0^\infty N(v)\, v\, dv}{\int_0^\infty N(v)\, dv} \qquad (10)$$

We note that the denominator of both expressions above is equal to N_0. The average velocity can be found using variable substitutions of

$$x = v^2 \qquad dx = 2v\, dv \qquad a = \frac{-m}{2kT} \qquad (11)$$

such that

$$
\begin{aligned}
v_{avg} &= \frac{1}{N_0}\int_0^\infty \frac{4\pi v^2\ N_0}{(2\pi k T/m)^{3/2}}\, e^{-mv^2/2kT}\ v\,dv = \frac{4\pi}{(2\pi k T/m)^{3/2}}\int_0^\infty v^2\ e^{-mv^2/2kT}\ v\,dv \\
&= \frac{4\pi}{(2\pi k T/m)^{3/2}}\int_0^\infty x\, e^{ax}\,\frac{dx}{2} = \frac{2\pi}{(2\pi k T/m)^{3/2}}\,\frac{e^{ax}}{a^2}\left(ax-1\right)\Big|_0^\infty \\
&= \frac{2\pi}{(2\pi k T/m)^{3/2}}\left[\frac{e^{-\infty}}{a^2}\left(\infty-1\right)-\frac{e^0}{a^2}\left(0-1\right)\right] = \frac{2\pi}{(2\pi k T/m)^{3/2}}\left(\frac{2kT}{-m}\right)^2 \\
&= \sqrt{\frac{8kT}{\pi m}}
\end{aligned}
\qquad (12)
$$

The average energy can be found in a like manner:

$$E_{avg} = \tfrac{3}{2}\,kT \qquad (13)$$

<u>In summary</u>: for the Maxwell-Boltzmann distribution, the most probable neutron velocity, $v_0 = v_p$, at 20°C (68°F) is 2200 m/s, which corresponds to a kinetic energy of E_0 = 0.0253 eV. Thermal-neutron cross-section data are tabulated at 2200 m/s or 0.0253 eV. The thermal neutron velocity, $v_T = v_p$, is the most probable velocity at temperature, T, and can be related to the most probable velocity at room temperature

$$v_T = \sqrt{\frac{2kT}{m}} = v_0\sqrt{\frac{T}{T_0}} \qquad (14)$$

This expression is useful in finding the thermal neutron velocity at temperatures other than 20°C. By comparing Eqs. (5) and (12), we note that the average neutron velocity, v_{avg}, and the most probable velocity, v_p, can be related via

$$v_{avg} = \frac{2}{\sqrt{\pi}}\,v_p \qquad (15)$$

The table below lists the average and most probable neutron velocities and energies.

Neutron	Energy	Velocity
Average	$E_{avg} = \frac{3}{2}kT$	$v_{avg} = \frac{2}{\sqrt{\pi}}v_p = 1.1284\,v_p$
Most Probable	$E_p = \frac{1}{2}kT$	$v_p = v_T = \sqrt{\frac{2kT}{m}}$ $E_T = \frac{1}{2}m v_p^2 = kT$

THERMAL NEUTRON FLUX

2200 meter-per-second Flux, ϕ_0

$$\phi_0 = n\, v_0 \tag{1}$$

Thermal Flux, ϕ_T

$$\phi_T = \int \phi(E)\, dE = n\, v_{avg} = \frac{2}{\sqrt{\pi}}\, n\, v_T \tag{2}$$

Beam vs. Reactor Flux

· relating the two different fluxes

$$\frac{\phi_0}{\phi_T} = \frac{n\, v_0}{\frac{2}{\sqrt{\pi}}\, n\, v_T} = \frac{\sqrt{\pi}}{2}\, \frac{v_0}{v_T} = \frac{\sqrt{\pi}}{2}\sqrt{\frac{E_0}{E_T}} = \frac{\sqrt{\pi}}{2}\sqrt{\frac{T_0}{T}} \tag{3}$$

where

$$T_0 = \frac{\frac{1}{2} m v_0^2}{k} = \frac{\frac{1}{2}(1.67492\times10^{-27}\,\text{kg})\,(2200\,\text{m/s})^2}{1.38066\times10^{-23}\ \text{J/°K}} = 293.6\text{°K} \tag{4}$$

Reaction Rate

$$F_a = \Sigma_a(E_0)\,\phi_0 = \overline{\Sigma}_{a,th}\ \phi_T \tag{5}$$

For 1/*v* absorbers, the absorption rate is independent of the neutron energy distribution. Non-1/*v* absorbers include: U-233, U-235, U-238, Pu-239, Cd, In, Xe-135, Sm-149.

Thermal Absorption Cross Section

$$\overline{\sigma}_{a,th} = \frac{\sqrt{\pi}}{2}\sigma_0\sqrt{\frac{T_0}{T}} = \frac{\sigma_{a,2200\,m/s}}{1.128}\sqrt{\frac{293\text{°K}}{T_{M,K}}} = \frac{\sigma_{a,2200\,\text{m/s}}}{1.128}\sqrt{\frac{0.0253\,\text{eV}}{E_{th}}} \tag{6}$$

Thermal Scattering Cross Section

$$\overline{\sigma}_{s,th} = \sigma_{s,2200\,\text{m/s}} \tag{7}$$